SOLUTION SHEET 12:

1. Recall that for a quartic $f(x) = x^4 + ax^3 + bx^2 + cx + d$ its resultant is given by $r(x) = x^3 - bx^2 + (ac - 4d)x - (a^2d + c^2 - 4bd)$
Let us denote the Galois group of f by G_f.

(i) $f(t,x) = x^3 + (2t+3)x - 1$ Lets compute the discriminant.
$\Delta(f) = -(4.(2t+3)^3 + 27)$. Note that $\Delta(f) \in \mathbb{Q}[t]$ which is a UFD. Now by Gauss' Lemma, $\Delta(f)$ is a square in $\mathbb{Q}(t)$ if and only if $\Delta(f)$ is a square in $\mathbb{Q}[t]$. However it can't be a square in $\mathbb{Q}[t]$ as $\Delta(f)$ has degree 3 in t.

(ii) $f(x) = x^4 - 7x^2 + 1$: Notice that reducing f mod 3 we obtain $\bar{f}(x) = x^4 + 2x^2 + 1$ which can be checked to be irreducible.

We compute its resultant to find $r(x) = x^3 + 7x^2 - 4x - 28$. Note that 2 is a root of $r(x)$ and $r(x)$ is not irreducible.

Computing the discriminant of $f(x)$ we get. $\Delta(f) = 16(49-4)^2 = 16.45^2 = (4.45)^2$ thus $\Delta(f)$ is a square. This shows that $G_f = V$.

(iii) $f(x) = x^4 - 5x + 2$: It can be seen by Eisenstein with 2 that f is irr.
The discriminant of f is $\Delta(f) = -27.(-5)^4 + 256(2)^3$ as it is negative $\Delta(f)$ is not a square. Let us compute its resultant
$r(x) = x^3 + (-5-8)x - 25 = x^3 - 13x - 25$ reducing $r(x)$ modulo 2 we get $\bar{r}(x) = x^3 + x + 1$ which is irreducible. This shows that $r(x)$ is irr. & $G_f = S_4$.

(iv) $f(x) = x^5 - x - 1$: Consider f as a polynomial over $\mathbb{F}_5$. Note that it has no roots in $\mathbb{F}_5$ thus by the Artin-Schreier thm. its irreducible. Now computing its discriminant we see that it is not a square, in fact $\Delta(f) = 19.151$.
Reducing f mod 2 we obtain $\bar{f}(x) = (1 + x + x^2)(1 + x^2 + x^3)$ then by Dedekind thm. G_f contains a cycle of type $(2,3)$. Moreover f is irreducible in $\mathbb{F}_5$ thus G_f contains a cycle of length 5. This shows that $G_f = S_5$.

(v) $f(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$: Notice that
$x^7 + 1 = (x-1) \cdot f(x)$. Thus $SF_{\mathbb{Q}}(f(x)) = SF_{\mathbb{Q}}(x^7+1)$.
Now as n is odd, $x^7 + 1 = -((-x)^7 - 1)$ thus the Galois group of $x^7 + 1$ is the same as the Galois group of $x^7 - 1$ by a change of variable $x \mapsto -x$. This shows that $G_f \cong \mathbb{Z}/6\mathbb{Z}$.

2. First, let us write the discriminant & resultent of $f(x) = x^4 + ax + b$.

$$\Delta(f) = -27a^4 + 256b^3$$
$$r(x) = x^3 - 4bx - a^2.$$

(i) $(a,b) = (1,1)$ In this case $x^4 + x + 1$ is irreducible as its reduction mod 2 is irr. moreover, $\Delta(f) = 256 - 27 = 229$ which is not a square. and $r(x) = x^3 - 4x - 1$ which is irreducible as its reduction mod 3 is $\bar{r}(x) = x^3 + 2x + 2$ and it doesn't admit any roots. This shows that $G_f = S_4$

(ii) $(a,b) = (8,12)$ $f(x) = x^4 + 8x + 12$. First note that to show f is irr. in $\mathbb{Q}[x]$ it suffices to show that is irr. in $\mathbb{Z}[x]$. It is not hard to see that $f(x)$ doesn't have a root in $\mathbb{Z}$. Moreover trying to write f as a product of two quadratic polynomials, we see that it doesn't work.

Now $\Delta(f) = 576^2$ and $r(x) = x^3 - 48x - 64$ reducing r mod 3 we get $\bar{r}(x) = x^3 - 2$ which doesn't have any roots $\Rightarrow r(x)$ is irr. $\Rightarrow G_f = A_4$.

(iii) $(a,b) = (3,3)$ $f(x) = x^4 + 3x + 3$, Eisenstein with 3 shows that $f(x)$ is irr. Now $\Delta(f) = -27 \cdot 3^4 + 256 \cdot 3^3 = (256 - 81) \cdot 3^3 = 175 \cdot 3^3 = 7 \cdot 5^2 \cdot 3^3$ is not a square. $r(x) = x^3 - 12x - 9$ as -3 is a root $r(x)$ is reducible and $G_f = D_4$ or C_4. Now to see that $G_f = D_4$ we show that f is irr. over $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{21})$. To see that first note that it can be factorized by a cubic as 3 doesn't divide $|G_f|$ in either case. It can be also seen that it can't be written as a product of the quadratic poly. by direct computation. To obtain the contradiction, use that $i, \sqrt{3} \notin \mathbb{Q}(\sqrt{\Delta})$.

(iv) $(a,b) = (5,5)$ as before $f(x)$ is irr. and $\Delta(f) = (256 - 135) \cdot 5^3 = 11^2 \cdot 5^3 \notin \mathbb{Q}^2$ We compute $r(x) = x^3 - 20x - 25$, as 5 is a root, $r(x)$ is reducible and $G_f = D_4$ or C_4. Now lets show that $f(x) = x^4 + 5x + 5$ is reducible over $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{5})$. It can be checked that

$$f(x) = \left(x^2 + \sqrt{5}x + \frac{5 - \sqrt{5}}{2}\right)\left(x^2 - \sqrt{5}x + \frac{5 + \sqrt{5}}{2}\right) \text{ and thus } G_f \cong C_4.$$

(v) $(a,b) = (0,1)$ then $f(x) = x^4 + 1$ in this case $\Delta(f) = 256 = 16^2$ and $r(x) = x^3 - 4x$ is reducible and $G_f = V$.